

Yukawa couplings in F-theory $SU(5)$

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1 Abstract

The fermion mass textures are discussed in the context of F-theory $SU(5)$ GUT. The tree-level up, down and charged lepton Yukawa couplings are computed in terms of the integrals of overlapping wavefunctions at the intersection points of three matter curves. All remaining entries in the fermion mass matrices can also be reliably estimated from higher order non-renormalizable Yukawa couplings mediated by heavy string modes and/or Kaluza-Klein states.

2 Wavefunction overlapping Integrals

In F-theory GUTs the trilinear Yukawa couplings are realised at the intersections of three matter curves where the zero-modes of two fermion fields and a Higgs boson reside. The structure of the zero-mode wavefunctions is found by solving the corresponding differential equations emerging from the twisted eight-dimensional Yang-Mills action [1]. The third generation up, down and charged lepton mass matrices are then computed in terms of the integrals of overlapping wavefunctions at the intersection point of three matter curves [2]. Furthermore, assuming that higher order non-renormalizable Yukawa couplings are generated through mediation of heavy string modes and/or Kaluza-Klein states, the calculation of all entries in the fermion mass matrices can be reduced to a similar computation. In fact, in this approach a rigorous and consistent application of the described method can be developed [3], by computing the relevant integrals for the NR-terms involving zero-mode and massive KK-mode overlapping wavefunctions. The complete calculation of all 3×3 Yukawa entries for the entire charged fermion mass spectrum shows that all the Yukawa coefficients are found within a reasonable range of values ($\leq \mathcal{O}(1)$) while the predicted fermion mass eigenvalues and Cabibbo Kobayashi Maskawa (CKM)-mixing in accordance to the experimental expectations. The tree-level Yukawa couplings are computed by the wavefunction overlap integrals

$$\lambda_{ij} = \int_S \psi_i \psi_j \phi dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2 \quad (1)$$

where S is the compact 4-d manifold wrapped by 7-brane surface associated to the gauge symmetry (here $SU(5)$). To compute the integral we use our knowledge of the wavefunctions' profile close to the intersection point. For localized solutions the zero-mode equations lead to a Gaussian profile of the general form

$$\psi \propto e^{-m^2 \frac{|q_1 z_1 + q_2 z_2|^2}{q}} \quad (2)$$

where m is a mass parameter related to a background Higgs vacuum expectation value (vev) $\langle \Phi \rangle$, $z_{1,2}$ complex coordinates on S and q_i are $U(1)$ charges ($q = \sqrt{q_1^2 + q_2^2}$). To make a reliable computation of

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the Yukawa couplings we also need a consistent normalization of the wavefunction. For a generic form of a wavefunction we have

$$\mathcal{C} = M_*^4 \int_S |\psi|^2 dz \wedge d\bar{z} = \pi \frac{M_*^4}{m^2 q} \mathcal{R}^2 \quad (3)$$

The factor $\pi \frac{M_*^4}{m^2 q}$ is the result of the gaussian integration along the coordinate normal to the curve. The factor \mathcal{R}^2 is introduced to account for the integration along the coordinate parametrising the curve. The mass scale m is naturally associated to the compactification scale $m \sim M_* \approx M_C$ on S , while the parameter \mathcal{R} ‘measures’ the integration along the matter curve inside the GUT surface S , thus clearly $\mathcal{R}^{-1} < M_C$ and one naturally expects that $\mathcal{R}^{-1} \approx M_{GUT}$. From renormalization group analysis we can see [4] that it is possible to obtain a scale independent normalization given by

$$\frac{1}{\sqrt{\mathcal{C}}} = \sqrt{\frac{q}{\pi}} e^{2/3(\mathcal{T}_{5/6} - \mathcal{T}_0)}$$

where the exponent $\mathcal{T}_{5/6} - \mathcal{T}_0$ is related to KK-massive mode threshold corrections expressed in terms of the analytic torsion \mathcal{T} [5]. The integral for three generic wavefunctions of the form (1) with arbitrary charges $q_i, q'_i, q''_i, i = 1, 2$ participating in the triple intersection the Yukawa couplings, gives [3]

$$\lambda = e^{2(\mathcal{T}_{5/6} - \mathcal{T}_0)} \frac{4\sqrt{\pi}}{q + q' + q''} \frac{(qq'q'')^{3/2}}{(q_1 q'_2 - q'_1 q_2)^2} \quad (4)$$

where $q''_i = -q_i - q'_i$ from charge conservation in the triple intersection, while $q = \sqrt{q_1^2 + q_2^2}$ etc.

2.1 Calculation of the bottom Yukawa coupling

In the present context we have assumed that our local F-theory GUT is $G_S = SU(5)$ with the top and bottom Yukawa couplings appearing in triple intersections of S with other seven branes. Locally the G_S symmetry is enhanced and in practice the Yukawa couplings should also be invariant under additional $U(1)$ symmetries. In the spectral cover approach we consider the $SU(5)$ embedding into the exceptional symmetry $E_8 \supset SU(5)_{GUT} \times SU(5)_\perp$ which is the highest symmetry in the elliptic fibration. We assume the background field configuration Φ with weights t_i along $SU(5)_\perp$ satisfying the tracelessness condition $\sum_{i=1}^5 t_i = 0$. We turn on a non-zero vev for Φ which breaks the symmetry down to $SU(5)_{GUT} \times U(1)^4$. At the intersection points where the Yukawa couplings are formed two appropriate linear combinations of these $U(1)$ ’s are preserved. The $SU(5)$ representations are characterised by various combinations of t_i ’s as shown in Table 1. An $SU(5)$ gauge invariant coupling must also comply with the cancellation of the t_i charges. In practice this means that for a given allowed coupling either the t_i must come in pairs with opposite sign, or their sum has to satisfy the traceless condition. The aim now is to calculate the top and bottom Yukawa couplings in a specific F-theory construction. Here the computation for the model [6, 3] where both up and down quark mass matrices are rank one in agreement with the hierarchical mass matrix structure is presented. In particular, we discuss the details for the bottom quark Yukawa coupling. In the model under consideration this is obtained from a trilinear term when families are accommodated according to $10^{(1)} \rightarrow \mathbf{10}_3, \bar{\mathbf{5}}^{(5)} \rightarrow \bar{\mathbf{5}}_3, \bar{\mathbf{5}}^{(2)} \rightarrow \bar{\mathbf{5}}_{h_d}$, and a Z_2 monodromy $t_1 \leftrightarrow t_2$ is assumed, so that

$$W_{tree} = \begin{array}{ccc} \mathbf{10}_3 & \cdot \bar{\mathbf{5}}_3 & \cdot \bar{\mathbf{5}}_{h_d} \\ t_2, & t_3 + t_5, & t_1 + t_4 \end{array}$$

where in the second line the t_i -combinations for the corresponding $SU(5)_{GUT}$ representations are shown. Clearly this $SU(5)$ invariant coupling satisfies the condition $\sum_i t_i = 0$ and therefore is allowed at tree-level. Recall now that the generators of the $U(1)$ s are given by the four diagonal generators, Q_i , of $SU(5)_\perp$. It is convenient to introduce the basis column vectors $|t_i\rangle_j = \delta_{ij}, i, j = 1, \dots, 5$. The local form of Q_i ’s

Field	$U(1)_i$	homology	$U(1)_Y$ -flux	$U(1)$ -flux
$10^{(1)}$	$t_{1,2}$	$\eta - 2c_1 - \chi$	$-N$	M_{10_1}
$10^{(2)}$	t_3	$-c_1 + \chi_7$	N_7	M_{10_2}
$10^{(3)}$	t_4	$-c_1 + \chi_8$	N_8	M_{10_3}
$10^{(4)}$	t_5	$-c_1 + \chi_9$	N_9	M_{10_4}
$5^{(0)}$	$-t_1 - t_2$	$-c_1 + \chi$	N	$M_{5_{hu}}$
$5^{(1)}$	$-t_{1,2} - t_3$	$\eta - 2c_1 - \chi$	$-N$	M_{5_1}
$5^{(2)}$	$-t_{1,2} - t_4$	$\eta - 2c_1 - \chi$	$-N$	M_{5_2}
$5^{(3)}$	$-t_{1,2} - t_5$	$\eta - 2c_1 - \chi$	$-N$	M_{5_3}
$5^{(4)}$	$-t_3 - t_4$	$-c_1 + \chi - \chi_9$	$N - N_9$	M_{5_4}
$5^{(5)}$	$-t_3 - t_5$	$-c_1 + \chi - \chi_8$	$N - N_8$	$M_{5_{hd}}$
$5^{(6)}$	$-t_4 - t_5$	$-c_1 + \chi - \chi_7$	$N - N_7$	M_{5_6}

Table 1 Matter curves available to accommodate the field representation content under $SU(5) \times U(1)_{t_i}$, their homology class and flux restrictions [6, 3]. A Z_2 monodromy $t_1 \leftrightarrow t_2$ is imposed.

is determined by demanding that the two of them annihilate the states participating in the vertex where the bottom Yukawa coupling is realized. In other words, for a given Yukawa coupling the fields involved will have zero eigenvalue for two combinations of the charge generators. Once we have identified them we easily construct the two orthogonal to them operators. Then the normalized charges of the fields are determined by the action of these operators on the corresponding states.

Recall now that the interacting fields for the bottom Yukawa have weights $t_{1,2}$, $t_{2,1} + t_4$ and $t_3 + t_5$. We first observe that the two operators

$$Q_3 = \frac{1}{\sqrt{2}}\{0, 0, 1, 0, -1\}, \quad Q_4 = \frac{1}{\sqrt{2}}\{0, 1, 0, -1, 0\} \quad (5)$$

annihilate the states participating in the bottom quark vertex

$$Q_{3,4}|t_{1,2}\rangle = 0, \quad Q_{3,4}|t_{1,2} + t_4\rangle = 0, \quad Q_{3,4}|t_3 + t_5\rangle = 0.$$

Thus, Q_1, Q_2 operators are part of a suitable $4d$ basis for vertices involving the above t_i -charge combinations. The most general vectors which fulfill $\sum t_i = 0$ and at the same time are orthogonal to them are

$$Q_1 = \frac{1}{\sqrt{2D}}\{-2(a+b), a, b, a, b\} \quad (6)$$

$$Q_2 = \frac{1}{\sqrt{10D}}\{2(a-b), 2a+3b, -3a-2b, 2a+3b, -3a-2b\} \quad (7)$$

with $D^2 = 3a^2 + 4ab + 3b^2$. For example, choosing $a = -b = 1$, then $Q_3 = \frac{1}{2}\{0, 1, -1, 1, -1\}$ and $Q_4 = \frac{1}{2\sqrt{5}}\{4, -1, -1, -1, -1\}$ corresponding to the $U(1)$ charge assignment obtained under the chain

$$SU(5)_\perp \rightarrow SU(4) \times SU(1) \rightarrow \cdots \rightarrow U(1)^4$$

Choosing $a = 0, b = 1$, we get $Q_3 = \frac{1}{2}\{-2, 0, 1, 0, 1\}$, $Q_4 = \frac{1}{\sqrt{30}}\{-2, 3, -2, 3, -2\}$ corresponding to another linear combination of the $U(1)$ charge assignment obtained under the breaking pattern $SU(5)_\perp \rightarrow SU(3) \times SU(2) \rightarrow \cdots \rightarrow U(1)^4$. Acting on the $|t_1\rangle$ and $|t_2 + t_4\rangle$ states with these operators we obtain the charges

$$\{q_1, q_2\} = \left\{-2\frac{a+b}{\sqrt{2D}}, 2\frac{a-b}{\sqrt{10D}}\right\}, \quad \{q'_1, q'_2\} = \left\{2\frac{a}{\sqrt{2D}}, 2\frac{2a+3b}{\sqrt{10D}}\right\} \quad (8)$$

Then, the quantities appearing in (4) are

$$q = \sqrt{q_1^2 + q_2^2} = \frac{2}{\sqrt{5}}, \quad q' = \sqrt{q_1'^2 + q_2'^2} = \sqrt{\frac{6}{5}}, \quad q_1' q_2 - q_1 q_2' = -\frac{2}{\sqrt{5}}$$

while q'' can be written in terms of q_i, q_i'' by charge conservation. We observe that the parameters involved in the computed integral are independent of a, b , and the specific breaking pattern of $SU(5)_\perp$. To compute the value of the overlapping wavefunctions integral and demonstrate the points discussed above, we proceed with the computation of the torsion \mathcal{T} in a simple case. As in ref [7] we take a line bundle $\mathcal{O}(n, -n)$ on a Hirzebruch surface $F_0 = P^1 \times P^1$. After some algebra we get [3]

$$\mathcal{T}_{5/6} - \mathcal{T}_0 = \zeta_0'(0) - \zeta_{-1}'(0) = -\frac{1}{2}$$

The resulting bottom quark coupling is $\lambda_b \approx 1.17$ while repeating the above analysis one finds $\lambda_t \approx 1.23$. Thus, in this simple example, the values of the third generation Yukawa couplings result to m_t, m_b masses which are close to the experimental findings. The remaining mass entries for the up and down are generated

Chiral Matter										
	M	N	Q	u^c	e^c		M	N	d^c	L
$10^{(1)}(F_3)$	1	0	1	1	1	$5^{(4)}(f_1)$	-1	0	-1	-1
$10^{(2)}(F_{2,1})$	1	-1	1	2	0	$5^{(1)}(\bar{f}_2)$	-1	0	-1	-1
$10^{(3)}(F_{1,2})$	1	1	1	0	2	$5^{(2)}(\bar{f}_3)$	-1	0	-1	-1
$10^{(4)}(-)$	0	0	0	0	0	$5^{(3)}(-)$	0	0	0	0

Higgs and Colour Triplets				
	M	N	T	$h_{u,d}$
$5^{(0)}(h_u, T)$	1	0	1	1
$5^{(5)}(h_d)$	0	-1	0	-1
$5^{(6)}(T)$	-1	1	-1	0

Table 2 The distribution of the chiral and Higgs matter content of the minimal model along the available curves, after the $U(1)_Y$ flux is turned on.

from non-renormalizable contributions. They emerge through the mediation of KK-massive modes and all Yukawa coefficients are calculable using the above procedure. Here we give a specific model with representation content given in Table 2 which is also capable of generating doublet-triplet splitting through the flux mechanism. Thus the up and down quark mass matrices have the form

$$M_u = \begin{pmatrix} \theta_{14}^2 \theta_{43}^2 & \theta_{14}^2 \theta_{43} & \theta_{14} \theta_{43} \\ \theta_{14}^2 \theta_{43}^2 & \theta_{14}^2 \theta_{43} & \theta_{14} \theta_{43} \\ \theta_{14} \theta_{43} & \theta_{14} & \sim 1 \end{pmatrix}, \quad M_d = \begin{pmatrix} \theta_{14}^2 \theta_{43}^2 & \theta_{14} \theta_{43}^2 & \theta_{14} \theta_{43} \\ \theta_{14}^2 \theta_{43} & \theta_{14} \theta_{43} & \theta_{14} \\ \theta_{14} \theta_{43} & \theta_{43} & \sim 1 \end{pmatrix}$$

where θ_{ij} represent $SU(5)$ singlet field vevs. It can be seen [3] that the above are in accordance with the quark mass ratios and CKM mixing. Moreover, a similar structure is obtained for the charged lepton mass matrix, whilst the neutrinos can be made consistent with tri-bi maximal mixing.

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